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NUMERICAL ANALYSIS OF THE DISTRIBUTION OF CURRENTS IN A PLASMA JET, WITH THE HALL EFFECT TAKEN INTO ACCOUNT

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Numerical analysis of the distribution of electric currents in a plasma medium, with the Hall effect taken into account, has been conducted in many works (see [1–3], for example) dedicated to studying intrachannel flows in various MHD devices.

The aim of the present paper is to study the peculiarities of the current distribution in stationary underexpanded plasma jets flowing into a neutral medium, with the magnetic field applied in such a manner as to be directed along the jet axis. This topic has been virtually neglected in the literature although its study would be of interest in designing and using certain types of MHD engines and accelerators. Although the problem seems to be somewhat simpler than that presented in [1–3] because of the absense of electrodes in the computational domain, it nevertheless has its own special features which are due to the fact that it has a free boundary and that the jet is not limited down the flow.

System of Equations and a Method of Solving It. We consider a stationary, axisymmetric jet of completely ionized plasma that flows into a nonconducting medium with a finite but fairly low pressure. It is assumed that the conductivity of the plasma is finite and is a function of electron temperature. The magnetic field applied is directed along the jet axis. To describe the jet, we use the magnetohydrodynamic approximation. Note that when no magnetic field is applied it is incorrect to use the continuous medium approximation for the jets flowing into a highly rarefied space, since at a distance from the source the mean free path of particles becomes comparable with the characteristic size of the jet. However, often the case is that the magnetic field strongly binds charged particles, and this presents an opportunity to obtain qualitatively correct results by making use of this approximation [4, 5].

Our mathematical model will be based upon the system of equations proposed in [6, 7] that has been derived from the general hydrodynamics equations of a plasma [8] under the assumption that the ionic gas is nondissipative and the energy transfer from electrons to ions is small because of the high mass ratio and low concentration of plasma. A model relation for electrons is used instead of the energy equation. Note that an adequate description of a distribution T_e in the jet is a complicated task and in order to solve it the capacity of the computers must be increased, for one needs to take into account not only heat conduction but also kinetic and emission processes whose influence may be considerable. However, for the purpose of qualitative analysis, we use the model relation (5), with our procedures based on [9, 10]. Note that using another model or an empirical relation instead of (5) does not alter the qualitative picture of the interaction of the plasma jet with the magnetic field.

If one takes as the main dimensional quantities the radius R_a of the jet initial cross section, the velocity V_a, and the density ρ_a of the plasma on the axis in the initial cross section, then the system of equations describing the behavior of magnetic fields and electric currents, with the Hall effect taken into account, may be written in the following dimensionless form:

$$\operatorname{div}(\rho \mathbf{V}) = 0; \quad (1)$$

$$\rho(\mathbf{V}\nabla)\mathbf{V} + \nabla(p_i + p_e) = [\mathbf{j}\,\mathbf{H}]; \quad (2)$$

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$$p_i = \text{const}\,\rho^{\gamma_i}; \tag{3}$$

$$p_e = \rho\, T_e; \tag{4}$$

$$T_e = \text{const}\,\rho^{\gamma_e - 1}; \tag{5}$$

$$\sigma = \text{const}\, T_e^{3/2}; \tag{6}$$

$$\mathbf{j} = \sigma\,(\mathbf{E} + [\mathbf{V}\,\mathbf{H}]) - \mu\,([\mathbf{j}\,\mathbf{H}] - \nabla\, p_e); \tag{7}$$

$$\text{rot}\,\mathbf{E} = 0; \tag{8}$$

$$\text{div}\,\mathbf{j} = 0; \tag{9}$$

$$\text{div}\,\mathbf{H} = 0; \tag{10}$$

$$\text{rot}\,\mathbf{H} = 4\,\pi\mathbf{j}. \tag{11}$$

Here, p_i and p_e are the ionic and electron pressure, respectively; γ_i is the adiabatic exponent for the ionic gas; γ_e is a parameter of the model relation for the electrons; $\mu = \beta\sigma/\rho$ is the mobility of electrons; $\mu\mathbf{H}$ is the Hall vector; $\beta = c_s m_i/\,(eR_a\sqrt{\rho_a}) = \sqrt{4\pi}\,\tilde{\beta}$ ($\tilde{\beta}$ is the exchange parameter [11]); $\mathbf{V} = (V_r, V_\varphi, V_z)$ is the plasma velocity vector; $\mathbf{H}{=}\mathbf{H}^0{+}\mathbf{H}^{\mathbf{i}}$ is the sum of the applied and induced magnetic fields; $\mathbf{H}^0 = (0, 0, H_z^0)$; the rest of the notation is conventional. The problem is considered in the cylindrical coordinate system (r, φ, z).

System (1)–(11) comprises equations of different types. Electromagnetic equations (8)–(11) are of elliptic type, while the type of the plasma dynamics equations (1), (2) depends on the mode of flow; when the flow is supersonic, they are of hyperbolic type. Thus, it makes sense to split the original problem into two subproblems, hydrodynamic and electromanetic, and then iterate them together until a consistent solution is obtained.

Consider the group of hydrodynamic equations (1), (2), with relations (3)–(5) taken into account. The difference between these equations and similar ones from [6, 7] lies in the presence of the Coriolis force and appearance of the azimuthal projection of the equation of motion that reflects the rotation of the jet around the axis of symmetry caused by the ponderomotive force.

Note that in traditional gas dynamics a system of equations similar to (1)–(5) is used for describing flows without shock waves. However, taking into consideration the relatively large degree of incomputability, the relatively large velocity of the outflow, the small slopes of the jet, and the fact that the magnetic field applied to the jet is of "viscous" kind [6], one can obtain qualitatively true results for jets with shock waves of low intensity.

To solve the group of hydrodynamic equations, we use the finite-difference march method with second order-of-magnitude accuracy in accordance with [6, 7]. Compared with [6, 7], there are additional boundary conditions. We let $V_\varphi = 0$ on the axis of the jet. To determine the velocity components on the free boundary of the jet, we add the azimuthal projection of the equation of motion. Written in the differential form, Bernoulli's equation along the boundary current line is

$$\frac{\partial}{\partial l}[(V_r^2 + V_z^2 + V_\varphi^2)/2] - \sin\theta\,(j_\varphi H_z - j_z H_\varphi)/\rho - \cos\theta\,(j_r H_\varphi - j_\varphi H_r)/\rho = 0,$$

where $\partial/\partial l$ is the derivative along the boundary current line; θ is the angle between the Oz axis and the tangent to the boundary current line at a given point.

Consider the group of electromagnetic equations (8)–(11), with the generalized Ohm's law taken into account. From Eqs. (7)–(9) one determines the vector of density of electric currents that flow in the jet under given values of the magnetic field and hydrodynamic parameters of the plasma. Given the values of the electric currents flowing in the jet, one determines the magnetic field from Eqs. (10) and (11). In this group of equations, we can also recognize two smaller problems — calculation of the currents and calculation of the induced magnetic field.

Consider the task of calculating the currents. In view of the axial symmetry, the process of solving Eqs. (7)–(9) can be reduced to solving one elliptic equation with mixed partial derivatives, from which j_r and

j_z are determined, and to calculating j_φ from the projection (7) onto the φ axis of the cylindrical coordinate system. The latter relation is of the form

$$j_\varphi = \sigma\,(V_z H_r - V_r H_z) - \mu\,(j_z H_r - j_r H_z).$$

Determination of j_r and j_z is based on the fact that the vector field j is solenoidal. We introduce a current function ψ, subject to the following conditions:

$$j_r = -\frac{1}{r}\frac{\partial \psi}{\partial z}; \tag{12}$$

$$j_z = \frac{1}{r}\frac{\partial \psi}{\partial r}. \tag{13}$$

Comparing (12) and (13) with (11), we note that $\psi = rH_\varphi^i/(4\pi)$.

Using relations (3) and (5), we write (7) as

$$\mathbf{E}' = [\mathbf{V}\,\mathbf{H}] - \mathbf{j}/\sigma - \beta/\rho\,[\mathbf{j}\ \ \mathbf{H}\,],$$

where

$$\mathbf{E}' = \nabla\,(\varphi_e + \beta\, c_e\, \gamma_e/\,(\gamma_e - 1)\,\rho^{\gamma_e - 1});$$

φ_e is the electrostatic potential; c_e is a constant in Eq. (5).

Applying the rot operation to the above relation and noting that $\operatorname{rot}\mathbf{E}' = 0$ and $E'_\varphi = 0$, one obtains the following equation from which the current function can be determined:

$$\frac{\partial}{\partial z}\Big(K^{11}\frac{\partial}{\partial z}\psi + K^{12}\frac{\partial}{\partial r}\psi + \frac{\mu H_\varphi^i}{r\sigma}\frac{\partial}{\partial r}\psi\Big) + \frac{\partial}{\partial r}\Big(K^{22}\frac{\partial}{\partial r}\psi + K^{21}\frac{\partial}{\partial z}\psi - \frac{\mu H_\varphi^i}{r\sigma}\frac{\partial}{\partial z}\psi\Big) = Q_1 + Q_2. \tag{14}$$

Here,

$$K^{11} = \Big(1 + (\mu H_z)^2\Big)/(r\sigma);$$

$$K^{12} = K^{21} = \mu^2 H_r H_z/(r\sigma);$$

$$K^{22} = (1 + (\mu H_r)^2)/(r\sigma);$$

$$Q_1 = \frac{\partial}{\partial z}(V_z H_\varphi - V_\varphi H_z) + \mu H_z (V_z H_r - V_r H_z);$$

$$Q_2 = \frac{\partial}{\partial r}(V_r H_\varphi - V_\varphi H_r) + \mu H_r (V_z H_r - V_r H_z).$$

Consider the boundary conditions for Eq. (14). Since the plasma jet is flowing into a nonconducting medium, the normal component of the electric current density vector equals zero at the free boundary of the jet. Therefore, the function ψ is constant there. Generally, the jet is unbounded down the flow. For the sake of definiteness, we limit the jet by some z_k. Let $\sigma = 0$ when $z > z_k$. Consequently, ψ is constant there, too. When there is no current flow through the initial cross section of the jet (there is no emission from the plasma source), ψ is also constant and equal to zero. On the axis, $\psi = 0$ by virtue of symmetry.

Thus, we have a boundary value problem of the first kind for the nonlinear elliptic equation (14) with mixed partial derivatives and with an operator that is not self-adjoint. Note, however, that terms due to which the operator is not self-adjoint have coefficients of the order μH, while the other terms have coefficients of the order $(\mu H)^2$ and 1. In the case where the azimuthal component of the Hall vector is not predominant, the terms with the coefficient μH will not exert a far-reaching influence on the solution, regardless of whether $\mu H \geqslant 1$ or $\mu H \leqslant 1$. Hence, they may be rearranged to the right-hand side of the equation, assuming they are known from the previous global iteration.

To obtain a numerical solution, it is convenient to map the physical region in coordinates (r, z) onto a

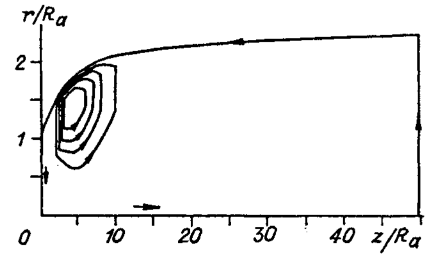

Fig. 1

rectangle or square (ξ, η). We introduce two nets in the calculation region, the main

$$S^1 = \left\{ \eta_{i_1} = i_1 h_\eta,\ \xi_{i_2} = i_2 h_\xi,\ 0 \leqslant i_1 \leqslant N,\ 0 \leqslant i_2 \leqslant M \right\}$$

and the auxiliary

$$S^2 = \left\{ \begin{array}{c} \eta_{i_1 - 1/2} = (i_1 - 1/2) h_\eta,\ \xi_{i_2 - 1/2} = (i_2 - 1/2) h_\xi, \\ 1 \leqslant i_2 \leqslant M,\ 1 \leqslant i_1 \leqslant N \end{array} \right\},$$

half-step shifted with respect to S^1. Based on the principles of [3, 12], an approximation of transformed equation (14) (with the above mapping taken into account) will yield a system of linear equations that can be effectively solved by using a locally optimal modification of the adjoint residual method presented in [13].

The components H_r^i and H_z^i of the magnetic field induced by the azimuthal current can be conveniently calculated on the basis of the potential theory in accordance with [7].

Different nets have been used in solving the subproblems. In calculating the hydrodynamic parameters of the plasma, a net with 101 points along the radial direction has been used. The components of the electric current density and the components of the induced magnetic field have been calculated on a net with 21 points along the radial direction and 101 points along the lengthwise direction. Interpolation has been used to switch from one net to the other.

Results of Numerical Analyses. Here we examine some characteristic results obtained from modeling a jet of argon plasma under conditions in which the induced magnetic field can be neglected. Let the radius R_a of the initial cross section of the jet be equal to 10 cm. The values of ionic and electron temperature, concentration, and velocity on the axis of the jet in the initial cross section are: $T_i = T_e = 0.5$ eV, $n_a = 10^{14}$ cm^{-3}, and $V_a = 6 \cdot 10^5$ cm/sec. Parameters $\gamma_i = 1.67$ and $\gamma_e = 1$. First we note that varying the parameters γ_i and $\gamma_e (1 < \gamma_e \leqslant \gamma_i)$ revealed no principal differences in the structure of the electric current distribution in the jet.

Under the given conditions, the degree of plasma ionization is rather small ($\approx 1.5 \cdot 10^{-3}$). But owing to the Ramsauer effect in the case of the argon plasma, one may notice that the collision frequency of charged particles with one another considerably dominates over that of charged particles with atoms. Using [14] for estimating the cross section of electron-atom collisions ($Q_{ea} \approx 0.7 \cdot 10^{-16}$ cm^2) gives $\nu_{ei}/\nu_{ea} \approx 6.4$. Therefore, in this case, the transfer of completely ionized plasma and electric charge transfer are of the same character.

Parameter profiles in the initial cross section of the jet have been determined from the model relations

$$\mathbf{F} = \exp(\mathbf{C} r^{l_1}), \quad r \in [0, 1], \quad l_1 = 1, 2, 3, \ldots,$$

$$V_r = \mathrm{const}\, V_z \sin\left(\frac{r\pi}{2}\right)^{l_2}, \quad l_2 = 1, 2, 3, \ldots,$$

where $\mathbf{F} = [\rho V_z]^{\mathrm{T}}$; $\mathbf{C} = [c_1, c_2]^{\mathrm{T}}$; c_1 and c_2 are constants ($c_1 < 0$ and $c_2 > 0$).

Consider a jet with ratio of the density on the axis in the initial cross section to the density of the external medium equal to 10. Varying the absolute value of the applied magnetic field strength vector within 25–125 Oe (while the initial values of the hydrodynamic parameters of the jet do not change), one

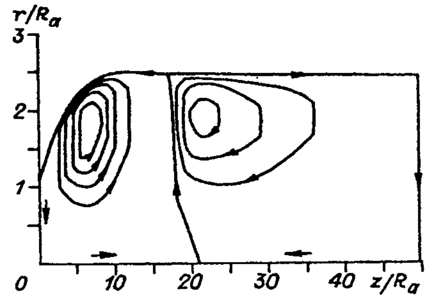

Fig. 2

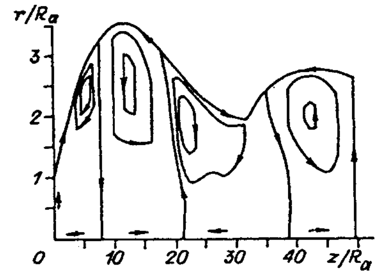

Fig. 3

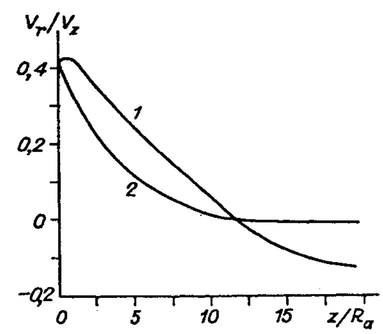

Fig. 4

obtains, in the case of supersonic flow, jets of various forms [6], including barrel-like ones. Relying on a posteriori estimates, we can point out that $\mathrm{Re}_m < 1$ in this situation, and the influence of the Hall effect on hydrodynamic parameters of plasma is insignificant. The only other thing to be mentioned here is that the jet starts rotating but its azimuthal speed is much lesser than its translational speed.

Consider a distribution of electric currents (components j_r and j_z of the current density vector) that arise in jets of different forms, with the Hall effect taken into account. Figures 1–3 show current lines when $H_z^0 = 125$, 75, and 25 Oe, respectively. The current lines were plotted so that equal parts of the total current flow between adjacent lines. This is so for each family of loops. The jet boundary is also a current line. The arrows show the direction of the current. The values of the Hall vector components in the rarefied part of the jet did not exceed 4.

Figure 1 shows that in this situation the electric current flows along the family of closed loops in one direction in the field of the jet. The current that flows between the adjacent lines is 1/5 of the total current. The maximum currents (j_r, j_z) flow near the jet boundary (where the current lines are condensed) in a region located at a certain distance from the initial cross section of the jet. Here, the rarefied part of the jet expands but the gradients of the hydrodynamic parameters are still sufficiently large. So are the azimuthal currents. The electric currents are small in the jet core and down the flow, since the parameters of the flow are distributed more uniformly.

When the jet has a barrel-like structure (rarefaction and condensation zones alternate), the distribution of j_r and j_z is qualitatively different (see Figs. 2 and 3). There appear regions in which the electric current flows along different families of loops in different directions. The families of these loops are separated by a line on which $H_\varphi^i = 0$ and on each side of the line this function has opposite signs. Zero surfaces do not coincide with the boundaries of the "barrels" but are shifted with respect to the latter. By analogy with [14], one can say that the so-called neutral (zero) surfaces appear in the field of the jet.

Analysis of Figs. 1–3 shows that the appearance and location of zero lines are connected with the appearance and location of the points of inflection on the curves that determine the jet boundary, i.e., one may conclude that the occurrence of zero lines depends on whether or not there is a change of sign of the flow acceleration in the radial direction. This conclusion is well illustrated by the appearance of a zero line near the jet initial cross section in Fig. 3. That the zero line has appeared here, is connected with the fact that the velocity vector has been artificially perturbed at the initial cross section, which ensures accelerating the flow in the radial direction (Fig. 4 shows the slope of the velocity vector along the jet boundary when $H_z^0 = 25$ and 75 Oe, represented by lines *1* and *2*, respectively).

Note also that when there is a barrel-like structure, a greater current flows in the first "barrel."

If we consider a greater ratio of the jet density on the axis in the initial cross section to the density of the surrounding medium $\rho_a/\rho_\infty \geqslant 100$ and a smaller absolute value of the applied magnetic field strength vector ($H_z^0 \leqslant 10$ Oe), taking care that the MHD interaction parameter on the jet boundary stays approximately the same, calculations show that, from the qualitative standpoint, the currents are distributed as in the case shown in Fig. 1. But because of the higher degree of plasma rarefaction, the analogous current lines are closer to the axis of symmetry and more extended along the z axis.

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